Application of the Variational Iteration Method to Nonlinear Volterra's Integro-Differential Equations

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Z. Naturforsch. **63a,** 538 – 542 (2008); received February 29, 2008

He's variational iteration method is proposed to solve nonlinear Volterra's integro-differential equations. The results reveal that this method is very effective and convenient in comparison with other methods.

Key words: Variational Iteration Method; Nonlinear Volterra's Integro-Differential Equations.

1. Introduction

He's variational iteration method [1, 2], which is a modified general Lagrange multiplier method [3], has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations which converge quickly to accurate solutions. It was successfully applied to autonomous ordinary differential equations [4], nonlinear partial differential equations with variable coefficients [5], Schrödinger-Korteweg-de Vries (KDV), generalized KDV and shallow water equations [6], Burgers' and coupled Burgers' equations [7], the linear Helmholtz partial differential equation [8] and recently to nonlinear fractional differential equations with Caputo differential derivative [9], and other fields [10 – 12]. Also, J. H. He used the variational iteration method for solving some integro-differential equations [13] by choosing the initial approximate solution in the form of an exact solution with unknown constants.

The aim of this paper is to extend the analysis of the variational iteration method to solve the general nonlinear Volterra's integro-differential equations of type

$$y'(x) = f(x) + \int_0^x k(x,t,y(t),y'(t))\mathrm{d}t, \qquad (1)$$

where $f(x)$ is a given continuous function and the unknown function $y(x)$ should be determined. In Section 2, the basic ideas of the variational iteration method are stated. Some examples are given in Section 3; also, we compare our results with other methods in this section. It is shown that this method is very simple and effective. A brief conclusion is given in Section 4.

2. Basic Ideas of the Variational Iteration Method

The basic concept of the variational iteration method [1, 2, 4, 5] and applications of it can be found in [14 – 20]. We consider the following general nonlinear system:

$$L[y(x)] + N[y(x)] = \psi(x),$$

where L is a linear operator, N a nonlinear operator and $\psi(x)$ a given continuous function. According to the variational iteration method, we can construct a correction functional in the form

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)[Ly_n(s) + N\tilde{y}_n(s) - \psi(s)]\mathrm{d}s,$$

where $y_0(x)$ is an initial approximation that satisfies the initial conditions, λ is a Lagrange multiplier which can be identified optimally via variational theory, the subscript n denotes the n-th approximation, and $\tilde{y}_n$ is considered as a restricted variation [1 – 3], i. e. $\delta\tilde{y}_n = 0$. It is shown that this method is very effective and easy for solving linear problems; its exact solution can be obtained by only one iteration, because λ can be exactly determined.

To solve (1) with the variational iteration method, we consider all terms as of restricted variation except for $y'(x)$. According to the variational iteration method, we define a correction functional as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left[y_n'(s) - f(s) - \int_0^s k(s,t,\tilde{y}_n(t),\tilde{y}_n'(t))\mathrm{d}t\right]\mathrm{d}s.$$

The stationary condition of the above correction functional can be expressed as

$$\lambda'(s) = 0,$$
$$1 + \lambda(s)|_{s=x} = 0.$$

The Lagrange multiplier, therefore, can be easily identified as

$$\lambda(s) = -1.$$

As a result, we obtain the iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \Big[y_n'(s) - f(s) - \int_0^s k(s,t,y_n(t),y_n'(t))\,\mathrm{d}t \Big]\mathrm{d}s. \quad (2)$$

3. Applications

In this section, we present some examples to show the efficiency and high accuracy of the variational iteration method for solving problem (1).

Example 1. Let us first consider Volterra's nonlinear integro-differential equation

$$y'(x) = 1 + \int_0^x y(t)y'(t)\,\mathrm{d}t$$

for $x \in [0,1]$, with the exact solution

$$y(x) = \sqrt{2}\tan\left(\frac{\sqrt{2}}{2}x\right).$$

According to (2) we have the iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \Big[y_n'(s) - 1 - \int_0^s y_n(t)y_n'(t)\,\mathrm{d}t \Big]\mathrm{d}s.$$

From the above integro-differential equation it is obvious that $y'(0) = 1$; therefore, we choose as the initial approximation $y_0(x) = x$. Then we obtain

$$y_1 = x + \frac{1}{6}x^3,$$

$$y_2 = x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \frac{1}{504}x^7,$$

$$y_3 = x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \frac{17}{2520}x^7 + \frac{19}{22680}x^9 + \frac{67}{831600}x^{11} + \frac{1}{196560}x^{13} + \frac{1}{7620480}x^{15},$$

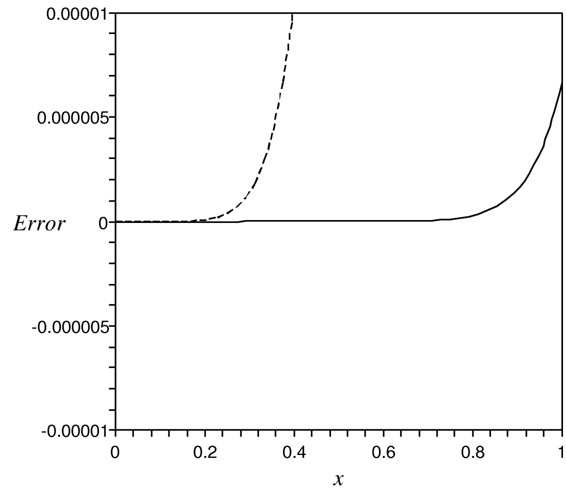

Fig. 1. The error of solutions obtained by the VIM and HPM. Solid line, VIM with 5 iterations; dashed line, HPM with 5 terms.

$$y_4 = x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \frac{17}{2520}x^7 + \frac{31}{22680}x^9 + \frac{571}{2494800}x^{11} + \frac{3331}{97297200}x^{13} + \frac{377017}{81729648000}x^{15} + \frac{761407}{1389404016000}x^{17} + \frac{12011}{210161952000}x^{19} + \frac{7439}{1471133664000}x^{21} + \frac{6338119}{17369184792960000}x^{23} + \frac{64237}{3089380694400000}x^{25} + \frac{25183}{28916603299584000}x^{27} + \frac{1}{43438564915200}x^{29} + \frac{1}{3600446356684800}x^{31},$$

and so on. It is obvious that the iterations converge to the power series of the exact solution. In order to show the efficiency and high accuracy of this method we present the curve of the absolute error of the N-th iteration, which is defined by

$$Ey_{N_{\mathrm{VIM}}}(x) = |\, y_{\mathrm{exact}}(x) - y_{N_{\mathrm{VIM}}}(x)\,|,$$
$$Ey_{N_{\mathrm{HPM}}}(x) = |\, y_{\mathrm{exact}}(x) - y_{N_{\mathrm{HPM}}}(x)\,|,$$

where VIM and HPM denote the variational iteration method and the homotopy perturbation method, respectively [21]. In Fig. 1, the error of the results obtained by the variational iteration method is presented; it is compared with the homotopy perturbation method results given in [21]. As seen from this figure, the solutions obtained by the present method are rather superior to those obtained by the HPM. Also, to perform the VIM is very simple.

Example 2. Consider the nonlinear integro-differential equation

$$y'(x) = -\frac{1}{2} + \int_0^x y'^2(t)\mathrm{d}t$$

for $x \in [0,1]$, with the exact solution

$$y(x) = -\ln\left(\frac{1}{2}x + 1\right).$$

According to (2) we have the following iteration formulation:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[y_n'(s) + \frac{1}{2} - \int_0^s [y_n'(t)]^2\mathrm{d}t\right]\mathrm{d}s.$$

From the above integro-differential equation it is clear that $y'(0) = -\frac{1}{2}$. Hence, we choose as an initial approximation $y_0(x) = -\frac{1}{2}x$ that satisfies it. We then obtain the approximations

$$y_1(x) = -\frac{1}{2}x + \frac{1}{8}x^2,$$

$$y_2(x) = -\frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{24}x^3 + \frac{1}{192}x^4,$$

$$y_3(x) = -\frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{24}x^3 + \frac{1}{64}x^4 - \frac{1}{240}x^5 + \frac{1}{1152}x^6 - \frac{1}{8064}x^7 + \frac{1}{129024}x^8,$$

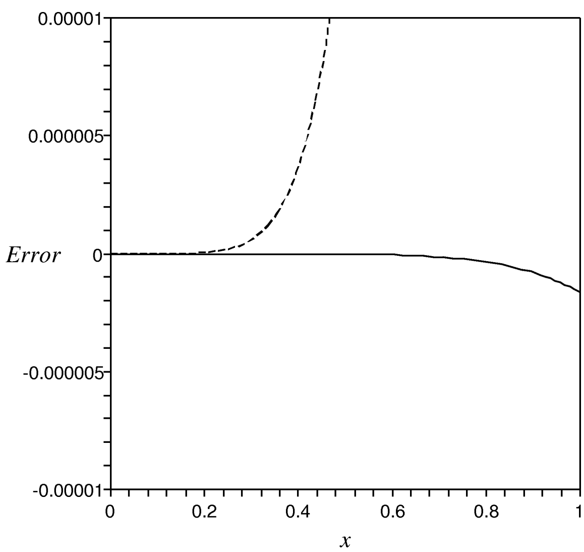

Fig. 2. The error of solutions obtained by the VIM and HPM. Solid line, VIM with 6 iterations; dashed line, HPM with 6 terms.

$$y_4(x) = -\frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{24}x^3 + \frac{1}{64}x^4 - \frac{1}{160}x^5 + \frac{13}{5760}x^6 - \frac{1}{1344}x^7 + \frac{29}{129024}x^8 - \frac{71}{1161216}x^9 + \frac{43}{2903040}x^{10} - \frac{1}{322560}x^{11} + \frac{5}{9289728}x^{12} - \frac{1}{13418496}x^{13} + \frac{1}{130056192}x^{14} - \frac{1}{1950842880}x^{15} + \frac{1}{62426972160}x^{16},$$

and so on. It is clear that the iterations converge to the exact solution. Also, in Fig. 2, we can see that the above solutions are better than the results obtained by the HPM [21].

Example 3. Consider the nonlinear integro-differential equation

$$y'(x) = -1 + \int_0^x y^2(t)\mathrm{d}t \tag{3}$$

for $x \in [0,1]$, with the boundary values $y(0) = 0$ [22]. Using the VIM (2) with the initial approximation $y_0(x) = -x$, that satisfies the boundary condition and $y'(0) = -1$, gives

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[y_n'(s) + 1 - \int_0^s y_n^2(t)\mathrm{d}t\right]\mathrm{d}s.$$

Then, we have the primary approximations

$$y_1(x) = -x + \frac{1}{12}x^4,$$

$$y_2(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{12960}x^{10},$$

$$y_3(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{6048}x^{10} - \frac{37}{7076160}x^{13} + \frac{109}{914457600}x^{16} - \frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{22},$$

$$y_4(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{6048}x^{10} - \frac{1}{157248}x^{13} + \frac{2663}{11887948800}x^{16} - \frac{4799}{677613081600}x^{19} + \frac{34109}{170758496563200}x^{22} - \frac{4507}{901604861853696}x^{25} + \frac{24354871}{221524314557453107200}x^{28}$$

$$
\begin{aligned}
&-\frac{1312457}{6288693911429529600000}x^{31} \\
&+\frac{253524431}{7647700951798842654720000}x^{34} \\
&-\frac{1709}{4053164656463290368000}x^{37} \\
&+\frac{241247}{5994301891937060782080000}x^{40} \\
&-\frac{1}{3913284129857696563200}x^{43} \\
&+\frac{1}{1246448394990169620480000}x^{46},
\end{aligned}
$$

and so on. The above example has been solved by Ghasemi et al. [22]. They showed that results obtained by the HPM are better in comparison with the results obtained by the wavelet-Galerkin method (WGM) [23]. The obtained solution by the HPM with 4 terms is

$$
y_{\mathrm{HPM}}(x)=\sum_{i=1}^{i=4}u_i(x)=-x+\frac{1}{12}x^4-\frac{1}{252}x^7+\frac{1}{6048}x^{10}. \tag{4}
$$

To show the effectiveness and advantages of the VIM, we present the residual of (3) by our solution, i. e. $y_4(x)$ and $y_{\mathrm{HPM}}(x)$, in Figure 3. As we can see the VIM with only four iterations is very close to the exact solution. Also, it is simple to apply this method.

Example 4. Consider the nonlinear Volterra's integro-differential equation [21]

$$
y'(x)=1+\int_0^x (x-t)\ln y'(t)\,\mathrm{d}t
$$

for $x\in[0,1]$. Using the VIM (2) with the initial approximation $y_0(x)=x$ and $y'(0)=1$ gives

$$
y_{n+1}(x)=y_n(x)-\int_0^x\left[y_n'(s)-1-\int_0^s(s-t)\ln y_n'(t)\,\mathrm{d}t\right]\mathrm{d}s.
$$

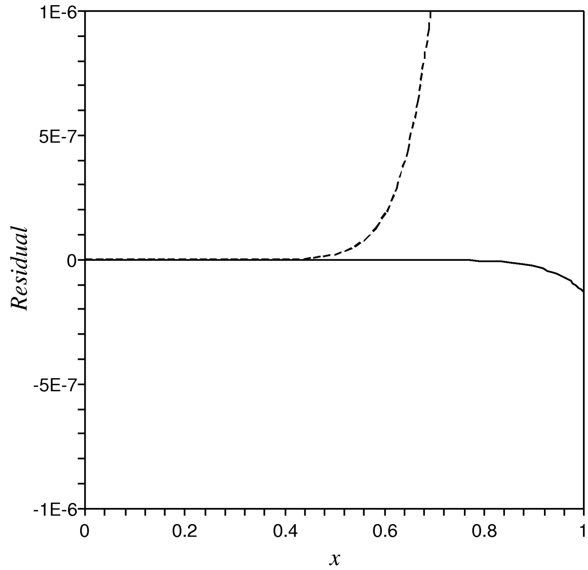

Fig. 3. The residual of solutions obtained by the VIM and HPM. Solid line, VIM with 4 iterations; dashed line, HPM with 4 terms.

From this, we obtain the following approximations:

$$
y_1(x)=y_2(x)=y_3(x)=\ldots=x.
$$

This is the exact solution.

4. Conclusions

We have studied the nonlinear Volterra's integro-differential equations with the variational iteration method. The initial approximation was chosen arbitraryly, although not in the form of the exact solution with unknown constants. The results showed that the variational iteration method is remarkably effective, and performing is very easy. In addition, it is more accurate than the homotopy perturbation method and the wavelet-Galerkin method.

Acknowledgements

The authors would like to thank the anonymous referees for their helpful comments.

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